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FUNCTIGRAPHS: AN EXTENSION OF PERMUTATION GRAPHS

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Abstract. Let G_1 and G_2 be copies of a graph G , and let $f: V(G_1) \rightarrow V(G_2)$ be a function. Then a functigraph $C(G, f) = (V, E)$ is a generalization of a permutation graph, where $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2), v = f(u)\}$. In this paper, we study colorability and planarity of functigraphs.

Keywords: permutation graph, generalized Petersen graph, functigraph

MSC 2010: 05C15, 05C10

1. INTRODUCTION AND DEFINITIONS

Throughout this paper, $G = (V(G), E(G))$ stands for a non-empty, simple and connected graph with order $|V(G)|$ and size $|E(G)|$. For a given graph G and $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph of G induced by S . The *distance*, $d(u, v)$, between two vertices u and v in G is the number of edges on a shortest path between u and v in G .

A graph G is *planar* if it can be embedded in the plane. A connected graph, with order at least 3, is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the exterior region [2]. A vertex v in a connected graph G is a *cut-vertex* of G if $G - v$ is disconnected. A *(vertex) proper coloring* of G is an assignment of labels, traditionally called colors, to the vertices of a graph such that no two adjacent vertices share the same color. A coloring using at most $k \geq 1$ colors is called a *(proper) k -coloring* and it is equivalent to the problem of partitioning the vertex set into k or fewer independent sets. The least number of colors needed to properly color a graph G is the chromatic number $\chi(G)$.

Following Chartrand and Harary (see p. 434 of [2]), a *permutation graph* P_α consists of two identical disjoint copies of a labeled graph G , say G_1 and G_2 , along

with $|V(G)| = n$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation α on $\{1, 2, \dots, n\}$. As noted by the authors, the graph $P_\alpha(G)$ depends not only on the choice of the permutation α but on the particular labeling of G as well.

For additional graph theory terminology we refer to [3]. We now introduce the study of a functigraph. We first recall that a *function graph* was independently introduced and was studied by Stephen Hedetniemi in [4] which overlaps our definition in a special case.

Definition 1.1. Let G_1 and G_2 be two copies of a graph G with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and let f be a function from $V(G_1)$ to $V(G_2)$. We define the functigraph $C(G, f)$ to be the graph that has the vertex set

$$V(C(G, f)) = V(G_1) \cup V(G_2),$$

and the edge set

$$E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2), v = f(u)\}.$$

We refer to $V(G_1)$ as the *domain* of the function f , to $V(G_2)$ as the *codomain* of f , and to $f(V(G_1))$ as the *range* of f .

Note that we use the notation $C(G, f)$ to refer to functigraphs for idiosyncratic reasons. A mnemonic is that the C reminds us that we have two *C*opies of G with function f mapping between them.

Observe that since G_1 and G_2 are copies of the same graph, the function f could be invertible. If so, then $C(G, f)$ is isomorphic to $C(G, f^{-1})$, which is a permutation graph. Indeed, functigraphs generalize permutation graphs [2]. That is, for any graph G , if f is a permutation on $V(G)$, then the functigraph $C(G, f)$ and the permutation graph $P_\alpha(G)$ coincide. The class of permutation graphs and thus the functigraphs include several interesting families of graphs such as

- (1) The prisms $C(C_n, f)$ where $f: V(G_1) \rightarrow V(G_2)$ is defined by

$$f(x) = \begin{cases} x + k & \text{if } 1 \leq x + k \leq n, \\ x + k - n & \text{if } x + k > n. \end{cases}$$

for $0 \leq k \leq n - 1$ (see (A) of Figure 1 when $k = 1$).

- (2) The Petersen graph $C(C_5, f)$ where $f: V(G_1) \rightarrow V(G_2)$ is defined by

$$f(x) = \begin{cases} 2x & \text{if } x = 1, 2, \\ 2x - 5 & \text{if } x = 3, 4, 5. \end{cases}$$

See (B) of Figure 1.

- (3) The hypercubes Q_n which are $C(Q_{n-1}, f(x) = x)$, $n \geq 1$.
- (4) $G \times K_2$, where G is any connected graph, in particular the ladder graphs $P_n \times P_2$ which are $C(P_n, f(x) = x)$.

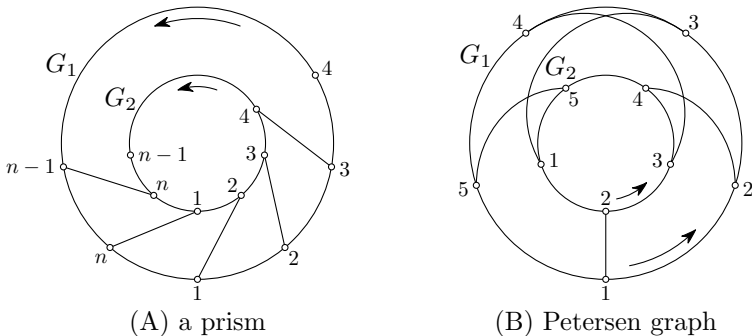


Figure 1. A prism and the Petersen graph

In this paper, we characterize the proper colorability and planarity in $C(G, f)$ when G is a cycle. In addition, we study colorability and planarity in $C(G, f)$ for an arbitrary graph G .

2. COLORABILITY OF FUNCTIGRAPHS

In this section we investigate the (proper) colorability of functigraphs. Clearly, $\chi(C(G, f)) \geq \chi(G)$, for all graphs G . Chartrand and Frechen [1] proved that, for every graph G and every permutation graph $P_\alpha(G)$ of G , $\chi(G) \leq \chi(P_\alpha(G)) \leq \lceil \frac{4}{3}\chi(G) \rceil$.

In this section, we generalize the result by Chartrand and Frechen for an arbitrary function $f: V(G_1) \rightarrow V(G_2)$. We begin our study by letting $G = C_n$ and then we proceed to arbitrary graphs. Let $C_n: v_1, v_2, \dots, v_n, v_1$ be a cycle of length n . For simplicity, we refer to a vertex by the index i of its label v_i ($1 \leq i \leq n$).

Proposition 2.1. *Let $G = C_n$ for n even. Then $2 \leq \chi(C(G, f)) \leq 3$. Moreover, for all $i, j \in V(G_1)$, $d(i, j)$ and $d(f(i), f(j))$ have the same parity if and only if $\chi(C(G, f)) = 2$.*

Proof. Let $G = C_n$ be an n -cycle for n even. Then $\chi(G_1) = \chi(G_2) = 2$, so the lower bound follows.

Now we consider the upper bound. Since G_2 is an even cycle, there is a 2-coloring of G_2 , say c , using colors 1 and 2. Let $S_t = \{x \in V(G_1): c(f(x)) = t\}$ for $t = 1, 2$. We next present a coloring of G , where vertices in S_1 are only colored 2 or 3, and

vertices in S_2 are colored 1 or 3. Let every other vertex in G_1 be colored 3, say these vertices form the set S . For $x \in S_1 - S$ let the color of x be 2, and if $x \in S_2 - S$ let the color of x be 1. Thus $\chi(C(G, f)) = 3$, giving the upper bound.

To see the characterization, note that if for all $i, j \in V(G_1)$, $d(i, j)$ and $d(f(i), f(j))$ have the same parity, then all cycles in $C(C_n, f)$ are even. This is the case if and only if $C(G, f)$ is bipartite, i.e. $\chi(C(G, f)) = 2$. For the converse, if there exist $i, j \in V(G_1)$ with $d(i, j)$ and $d(f(i), f(j))$ having different parities, then $C(G, f)$ contains an odd cycle, and so $\chi(C(G, f)) \geq 3$. \square

For an odd cycle C_n , we have the following:

Proposition 2.2. *Let $G = C_n$ for n odd. Then $3 \leq \chi(C(G, f)) \leq 4$. Moreover, $\chi(C(G, f)) = 4$ if and only if f is a constant function.*

Proof. Since $C(G, f)$ contains an odd cycle, $\chi(C(G, f)) \geq 3$. To obtain the upper bound, we show that a coloring with at most 4 colors can be found, by considering two cases.

Case 1. f is a constant function. Say $f: V(G_1) \rightarrow V(G_2)$ given by $f(x) = a$ for all $x \in V(G_1)$ and for some $a \in V(G_2)$ with $1 \leq a \leq n$. Define a coloring $c_1: V(C(G, f)) \rightarrow \{1, 2, 3, 4\}$ by 3-coloring G_1 with colors 1, 2, and 3, color the vertex labeled a by 4, and properly color the rest of G_2 with two colors 1 and 2. Thus $\chi(C(G, f)) \leq 4$.

Case 2. f is not a constant function. Since G_2 is an odd cycle, there exists a 3-coloring of $V(G_2)$, say c , using colors 1, 2, and 3. Without loss of generality, color G_2 so that color 3 is only used once, and it is used on a vertex of G_2 of minimum degree in $C(G, f)$. Let $S_t = \{x \in V(G_1): c(f(x)) = t\}$ for $t = 1, 2, 3$. Thus $|S_3| \in \{0, 1\}$. We consider two subcases.

Subcase 2.1. $|S_3| = 0$. First we assume that either $S_1 = \emptyset$ or $S_2 = \emptyset$, say the former. Then $V(G_1) = S_2$ and one can properly color G_1 with colors 1, 3, and 4. Thus $\chi(C(G, f)) \leq 4$. Next we assume that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Then find two consecutive vertices in G_1 , say k and $k + 1$ such that $k \in S_1$ and $k + 1 \in S_2$, or $k \in S_2$ and $k + 1 \in S_1$, say the former. Then complete the 3-coloring c_2 of $C(G, f)$ by coloring every other vertex of the cycle G_1 starting with $k + 2$ with color 3, say these vertices form the set S . For $i \in V(G_1) - S$ let the color of i be

$$c_2(i) = \begin{cases} 2 & \text{if } i \in S_1, \\ 1 & \text{if } i \in S_2. \end{cases}$$

And so $\chi(C(G, f)) = 3$ in this case.

Subcase 2.2. $|S_3| = 1$. Then let $V(G_1) \cap S_3 = \{k\}$. Then either $k + 1 \in S_1$ or $k + 1 \in S_2$, say the former. Now define a coloring $c_2: V(C(G, f)) \rightarrow \{1, 2, 3\}$ by

letting every other vertex in G_1 be colored 3 starting with $k + 1$, say these vertices form the set S . For $x \in S_1 - S$ let the color of x be 2, if $x \in S_2 - S$ let the color of x be 1, and the color of k is

$$c_2(k) = \begin{cases} 1 & \text{if } k - 1 \in S_1, \\ 2 & \text{if } k - 1 \in S_2. \end{cases}$$

And so, $\chi(C(G, f)) = 3$.

We now prove the characterization. Assume that $f(x) = a$ for all $x \in V(G_1)$ and for some a with $1 \leq a \leq n$. Since the vertex labeled a in G_2 is adjacent to every vertex of the odd cycle G_1 , we have that $\chi(C(G, f)) \geq 4$, and by the proof above we have that $\chi(C(G, f)) \leq 4$, thus $\chi(C(G, f)) = 4$.

For the converse, assume to the contrary that f is not a constant function. Then the coloring c_2 above proves that $\chi(C(G, f)) = 3$, a contradiction. \square

We now give bounds for the chromatic number of the functigraph $C(G, f)$ in terms of the chromatic number of the graph G , for any graph G and for all functions f . The upper bound of the theorem below is a special case of a result independently proved by Hedetniemi in [4].

Theorem 2.3. *If $\chi(G) = \alpha$, then $\alpha \leq \chi(C(G, f)) \leq \alpha + \lceil \frac{1}{2}\alpha \rceil$. Both bounds are sharp.*

Proof. Since $C(G, f)$ contains a copy of G , $\chi(C(G, f)) \geq \alpha$. For the upper bound let G_1 and G_2 be the two copies of G in $C(G, f)$. Let c^* be a coloring of G_2 with the color classes $1, 2, \dots, \alpha$ such that $W_i = \{w \in V(G_2) : c^*(w) = i\}$ for $1 \leq i \leq \alpha$ with $V(G_2) = \bigcup_{i=1}^{\alpha} W_i$. And let $S_i = \{v \in V(G_1) : c^*(f(v)) = i\}$ for $1 \leq i \leq \alpha$. Since G_1 is also α -partite, it follows that $V(G_1) = \bigcup_{i=1}^{\alpha} U_i$, where U_i is the independent set corresponding to W_i ($1 \leq i \leq \alpha$). We construct the coloring c of $C(G, f)$ by $c: V(C(G, f)) \rightarrow \{1, 2, \dots, \lceil \frac{3}{2}\alpha \rceil\}$, where

$$c(v) = \begin{cases} i & \text{if } v \in W_i \ (1 \leq i \leq \alpha), \\ j + \lceil \frac{1}{2}\alpha \rceil & \text{if } v \in U_j \cap S_j \ (1 \leq j \leq \lfloor \frac{1}{2}\alpha \rfloor), \\ j & \text{if } v \in U_j - S_j \ (1 \leq j \leq \lfloor \frac{1}{2}\alpha \rfloor), \\ 2\alpha + 1 - k & \text{if } v \in U_k \ (\lfloor \frac{1}{2}\alpha \rfloor + 1 \leq k \leq \alpha). \end{cases}$$

And so $\chi(C(G, f)) \leq \alpha + \lceil \frac{1}{2}\alpha \rceil$.

To see the sharpness of the lower bound, let G be an α -partite graph with $V(G) = \{V_1, V_2, \dots, V_\alpha\}$ being a partition of $V(G)$ into independent sets, and f be the identity

function. Let G_1 and G_2 be the two copies of G , with $V(G_1) = \{V_1^1, V_2^1, \dots, V_a^1\}$ and $V(G_2) = \{V_1^2, V_2^2, \dots, V_a^2\}$ be the corresponding partitions into independent sets of $V(G_1)$ and $V(G_2)$. Then define a coloring $c: V(C(G, f)) \rightarrow \{1, 2, \dots, a\}$ given by

$$c(v) = \begin{cases} i & \text{if } v \in V_i^1 \ (1 \leq i \leq a), \\ j + 1 & \text{if } v \in V_j^2 \ (1 \leq j \leq a - 1), \\ 1 & \text{if } v \in V_a^2. \end{cases}$$

Thus $\chi(C(G, f)) = \chi(G) = a$.

To see the sharpness of the upper bound, let (1) G be a complete a -partite graph $K_{a,a,\dots,a}$, (2) G_1 be a copy of G with the partition of $V(G_1)$ into independent classes as $V(G_1) = \bigcup_{i=1}^a U_i$, where $U_i = \{u_{i1}, u_{i2}, \dots, u_{ia}\}$, (3) G_2 be another copy of G with the partition of $V(G_2)$ into independent classes as $V(G_2) = \bigcup_{j=1}^a W_j$, where $W_j = \{w_{j1}, w_{j2}, \dots, w_{ja}\}$, and (4) $f(u_{ij}) = w_{jj}$, for all $i, j, 1 \leq i, j \leq a$.

Then define the coloring $c: V(C(G, f)) \rightarrow \{1, 2, \dots, \lceil \frac{3}{2}a \rceil\}$ given by

$$c(v) = \begin{cases} j & \text{if } v \in W_j \ (1 \leq j \leq a), \\ i + \lceil \frac{1}{2}a \rceil & \text{if } v = u_{ij} \ (i = j, 1 \leq i \leq \lfloor \frac{1}{2}a \rfloor), \\ i & \text{if } v = u_{ij} \ (i \neq j, 1 \leq i \leq \lfloor \frac{1}{2}a \rfloor), \\ 2a + 1 - i & \text{if } v \in U_i \ (\lfloor \frac{1}{2}a \rfloor + 1 \leq i \leq a). \end{cases}$$

Thus $\chi(C(G, f)) \leq \lceil \frac{3}{2}a \rceil$. We claim that $\chi(C(G, f)) = \lceil \frac{3}{2}a \rceil$. Assume, to the contrary, that $\chi(C(G, f)) = l < \lceil \frac{3}{2}a \rceil$. Then $l < \frac{3}{2}a$. Since we have that $\langle \{w_{11}, w_{22}, \dots, w_{aa}\} \rangle \cong K_a$, we then obtain $\chi(\langle \{w_{11}, w_{22}, \dots, w_{aa}\} \rangle) = a$. Without loss of generality, we assign $c(w_{jj}) = j$ for $1 \leq j \leq a$. Suppose that r of the sets U_i have the same color assigned to each vertex of set. Since each vertex of U_i ($1 \leq i \leq a$) is adjacent to a different vertex of the clique $\{w_{11}, w_{22}, \dots, w_{aa}\}$, it follows that the r colors must be distinct from the a colors already used. Then $\chi(C(G, f)) \geq a + r$. Since at least two colors are used for each of the remaining $a - r$ sets of U_i , $r + 2(a - r) \leq l$, and so $r \geq 2a - l$. Therefore, $\chi(C(G, f)) \geq a + r \geq a + 2a - l = 3a - l > 3a - \frac{3}{2}a = \frac{3}{2}a > l = \chi(C(G, f))$, which is a contradiction. \square

Below is an example that shows the construction of the coloring of $C(G, f)$ for $G = K_{3,3,3}$ for the sharpness of the the upper bound above.

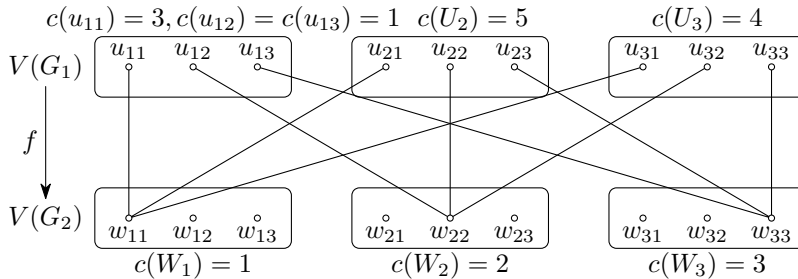


Figure 2. An example of $\chi(C(K_{3,3,3}, f)) = 5$

2. PLANARITY OF FUNCTIGRAPHS

Chartrand and Harary [2] proved a result analogous to Kuratowski's theorem for outerplanar graphs: a connected graph G is outerplanar if and only if it contains no subgraph which is homeomorphic to K_4 or $K_{2,3}$. Further, Chartrand and Harary characterized planar permutation graphs that contain no cut-vertices. The result states that the permutation graph $P_\alpha(G)$ of a nonseparable graph G is planar if and only if G is outerplanar and α is *dihedral*. (See Ch. 4 of [5] for the dihedral groups.)

We begin this section with a characterization of planar functigraphs $C(C_n, f)$. We further characterize planar functigraphs $C(G, f)$ for an arbitrary graph G , and thus generalize the results obtained by Chartrand and Harary on permutation graphs to functigraphs.

We first consider $G = C_n$. Let $C_n: v_1, v_2, \dots, v_n, v_1$ be a cycle of length n . For simplicity, we refer to each vertex of the cycle by its index i of its label v_i ($1 \leq i \leq n$). Let G_1 and G_2 be already embedded in the plane, with both labeled counterclockwise as in (A) of Figure 3. Let $f: V(G_1) \rightarrow V(G_2)$ be a function and let $\sigma: V(G_2) \rightarrow V(G'_2)$ be an identity function, where G'_2 is a copy of G_2 with clockwise orientation. Then the composition $g := \sigma \circ f$ maps (B) of Figure 3 to (A) of Figure 3. Thus we only need to consider when $V(G_1)$ and $V(G_2)$ are labeled counterclockwise as in (A) of Figure 3.

For the rest of the paper, the function f such that $f(i) = a_i$ for all $i = 1, 2, \dots, n$ will be denoted by $f = (a_1, a_2, \dots, a_n)$. Note that if all a_i 's are distinct and $\bigcup_{i=1}^n \{a_i\} = \bigcup_{j=1}^n \{j\}$, then $f: V(G_1) \rightarrow V(G_2)$ is a bijection and thus $C(G, f)$ is a permutation graph.

Example. Let $G = C_5$ be a cycle of length 5. Let G_1 and G_2 be copies of G with labelings of vertices assigned as in Figure 4. Then $f = (2, 2, 3, 3, 1)$ means

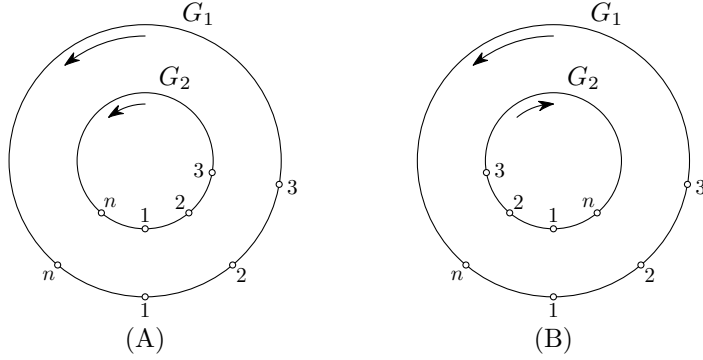


Figure 3. Labelings of $V(G_1)$ and $V(G_2)$

that $f(1) = 2$, $f(2) = 2$, $f(3) = 3$, $f(4) = 3$, and $f(5) = 1$. Note that $C(G, f)$ is isomorphic to $C(G, \tilde{f})$ for $\tilde{f} = (1, 1, 2, 2, 5)$.

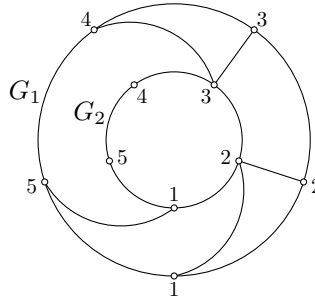


Figure 4. $f_1: V(G_1) \rightarrow V(G_2)$ as in Example

Notice that, by allowing relabeling, we can assume that there exists a vertex in G_1 that is mapped to vertex n in G_2 . If $f(n) = a_n \neq n$ then there exists an automorphism $\bar{\sigma}: i \rightarrow i + n - a_n$ such that $\bar{\sigma} \circ f(n) = n$.

We next characterize planar functigraphs when G is a cycle. One can easily check that $C(C_3, f)$ is planar for any function f : there are three distinct (non-isomorphic) cases. Thus, we consider $G = C_n$ for $n \geq 4$.

We define a function f of a functigraph $C(C_n, f)$ to be *semi-monotonic*, and we denote it by $f \in SM_n$, if and only if $f = (a_1, a_2, \dots, a_n)$ satisfies

- (1) there exists $k \in V(G_1)$ ($1 \leq k \leq n$) such that

$$1 \leq a_k \leq a_{k+1} \leq \dots \leq a_{n-1} \leq a_n \leq a_1 \leq a_2 \leq \dots \leq a_{k-1} \leq n$$

or

(2) there exists $k \in V(G_1)$ ($1 \leq k \leq n$) such that

$$1 \leq a_k \leq a_{k-1} \leq \dots \leq a_2 \leq a_1 \leq a_n \leq a_{n-1} \leq \dots \leq a_{k+1} \leq n.$$

A graph H is called a *subdivision* of a graph G if one or more vertices of degree 2 are inserted into one or more edges of G (see p. 236 of [3]). Now we recall Kuratowski's Theorem: a graph G is planar if and only if G does not contain K_5 , $K_{3,3}$, or a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Proposition 3.1. *Let $G = C_n$ be a cycle of length $n \geq 4$. Let G_1 and G_2 be copies of G with cyclic labelings. Without loss of generality, we assume that $f(n) = n$. Further, by relabeling the vertices so that two adjacent vertices 1 and n in G_1 get mapped to two different vertices in G_2 , we may assume that $f(1) \neq n$ if f is not a constant function. Then $C(G, f)$ is planar if and only if f is semi-monotonic (f could be a constant function).*

Proof. (\Leftarrow) It is easy to check.

(\Rightarrow) Let $C(G, f)$ be planar for $G = C_n$ with $n \geq 4$. Assume, to the contrary, that f is not semi-monotonic. Then, without loss of generality, we may assume that there exist vertices r, s, t, u in G_1 such that $1 \leq r < s < t < u \leq n$ and $1 \leq a_r \leq a_t < a_s \leq a_u \leq n$. We consider three cases.

Case 1. $|f(V(G_1))| = 2$: Note that $C(G, f)$ contains a subdivision of $H_1 \cong K_{3,3}$ (see Figure 5) as a subgraph.

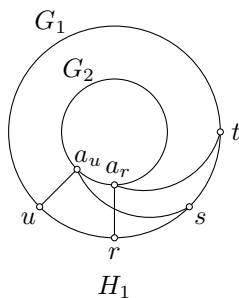


Figure 5

Case 2. $|f(V(G_1))| = 3$: Notice that $C(G, f)$ contains a subdivision of either H_1 , H_2 , or H_2^* as a subgraph. Moreover, H_2 also contains a subdivision of $K_{3,3}$ as a subgraph, where the two bipartite sets are $\{r, t, a_s\}$ and $\{s, u, a_r\}$. Also note that H_2^* contains a subdivision of $K_{3,3}$ as a subgraph, where the two bipartite sets are $\{r, t, a_u\}$ and $\{s, u, a_t\}$ (see Figure 6).

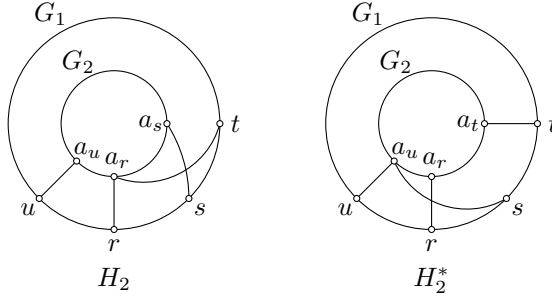


Figure 6

Case 3. $|f(V(G_1))| \geq 4$: Notice that $C(G, f)$ contains a subdivision of either H_1 , H_2 , H_2^* , or H_3 as a subgraph. Moreover H_3 contains a subdivision of $H'_3 \cong K_{3,3}$ as a subgraph, where the two bipartite sets are $\{t, a_r, a_s\}$ and $\{s, a_t, a_u\}$ (see Figure 7).

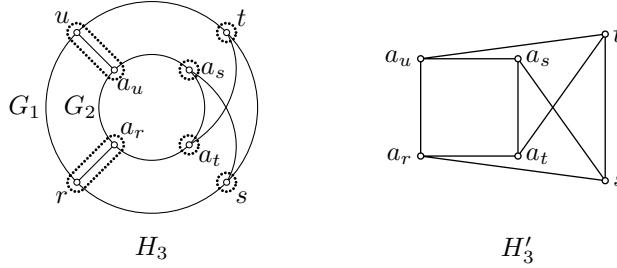


Figure 7

Thus, by Kuratowski's Theorem, $C(G, f)$ is not planar, which is a contradiction to the planarity hypothesis. \square

The following is an immediate result of Proposition 3.1. The sufficient direction of the corollary below, for any graph G , was independently proved by Hedetniemi in the study of function graph in [4].

Corollary 3.2. *Let G be a graph without cut-vertices. Then the functigraph $C(G, f)$ is planar if and only if G is outerplanar and f is semi-monotonic (f could be a constant function).*

Proof. (\Leftarrow) It is easy to check.

(\Rightarrow) Assume, to the contrary, that either G is not outerplanar or f is not semi-monotonic. If G is not outerplanar, then there is a vertex $v \in V(G)$ such that v doesn't lie on the exterior region. Then the edge $vf(v)$ will cross one of the edges of G , a contradiction to the planarity hypothesis of $C(G, f)$. Thus G is outerplanar, and we can assume that every nonseparable outerplanar graph G is cyclically labeled. If $f \notin SM_n$, then $C(G, f)$ is not planar, a contradiction. \square

Remark. The condition of G having no cut-vertex is necessary. To see this, let $G = P_6$ be a path of length 5, with cut-vertices, as in Figure 8. Define $f: V(G_1) \rightarrow V(G_2)$ by

$$f(x) = \begin{cases} \frac{x+6}{2} & \text{if } x \text{ is even,} \\ \frac{x+1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

Then $C(G, f)$ is planar and G is outerplanar, but $f = (1, 4, 2, 5, 3, 6)$ is not semi-monotonic.

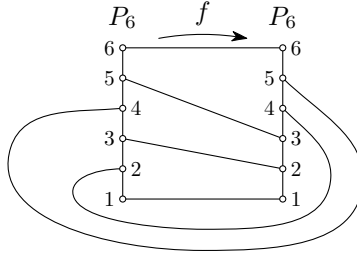


Figure 8. $C(P_6, f)$ for the Remark above

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